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On some fractional generalizations of the Laguerre polynomials and the Kummer function

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Dedicated to Professor Ivan Dimovski, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, on the occasion of his 75th Birth Anniversary.

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ABSTRACT

This paper refers to some generalizations of the classical Laguerre polynomials. By means of the Riemann–Liouville operator of fractional calculus and Rodrigues' type representation formula of fractional order, the Laguerre functions are derived and some of their properties are given and compared with the corresponding properties of the classical Laguerre polynomials. Further generalizations of the Laguerre functions are introduced as a solution of a fractional version of the classical Laguerre differential equation. Likewise, a generalization of the Kummer function is introduced as a solution of a fractional version of the Kummer differential equation. The Laguerre polynomials and functions are presented as special cases of the generalized Laguerre and Kummer functions. The relation between the Laguerre polynomials and the Kummer function is extended to their fractional counterparts.

1. Preliminaries

Fractional calculus is one of the most intensively developing areas of mathematical analysis. Its fields of application range from biology through physics and electrochemistry to economics, probability theory and statistics. On behalf of the nature of their definition the fractional derivatives provide an excellent instrument for the modeling of memory and hereditary properties of various materials and processes. Half-order derivatives and integrals prove to be more useful for the formulation of certain electrochemical problems than the classical methods [1]. Fractional differentiation and integration operators are also used for extensions of the diffusion and wave equations [2,3] and, recently, of the temperature field problem in oil strata [4]. In special treaties (as in [5–8]) the mathematical aspects and applications of the fractional calculus are extensively discussed.

In this paper the Laguerre functions are derived by Rodrigues' type representation formula for the classical Laguerre polynomials, generalized by means of the Riemann–Liouville fractional differentiation operator. The generalized Kummer function is obtained by a modified power series method as a solution of the fractional extension of the Kummer differential equation. By a similar approach the generalized Laguerre function is introduced.

For our purposes we adopt in this paper the **Riemann–Liouville fractional derivative** of $f(t)$ of order μ, defined by

$$D^{\mu}f(t) \equiv D^{m}\left[J^{m-\mu}f(t)\right],$$

where $m \in \mathbb{N}$, $m-1 \le \mu < m$, and

$$J^{m-\mu}f(t) \equiv \frac{1}{\Gamma(m-\mu)}\int_0^t (t-\tau)^{m-\mu-1}f(\tau)\mathrm{d}\tau$$

is the Riemann–Liouville fractional integral of $f(t)$ of order $m-\mu$.

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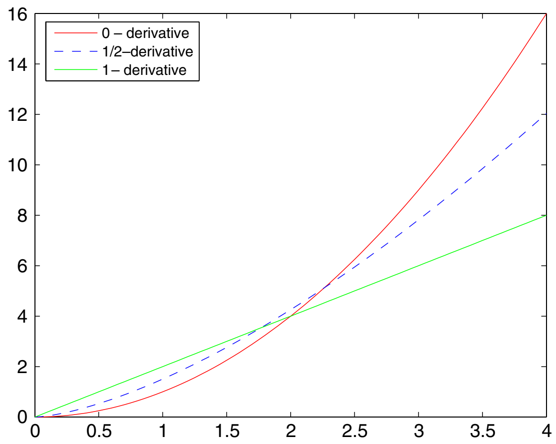

Fig. 1. 1/2-fractional derivative of $f(t) = t^2$.

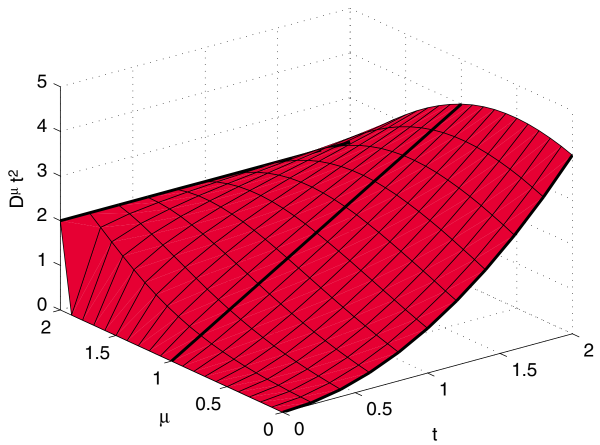

Fig. 2. Fractional derivatives of $f(t) = t^2$ of order $0 \le \mu \le 2$ on the interval $[0, 2]$.

In comparison to the classical calculus let us mention that, for example, if $\mu \ge 0$, $t > 0$ and $\alpha > -1$, then the fractional derivative of the power function t^α is given by

$$D^\mu t^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\mu+1)} t^{\alpha-\mu}. \tag{1}$$

Fig. 1 illustrates the 1/2-fractional derivative of $f(t) = t^2$. The three-dimensional graph of the fractional derivatives of $f(t) = t^2$ of order $0 \le \mu \le 2$ on the interval $[0, 2]$ is given on Fig. 2. These figures visualize the fact that the fractional derivatives are ''continuously'' distributed between the standard integer-order derivatives. Another illustration of the continuity of the fractional differential operator is provided on Fig. 3. It shows the changes of $D^\mu t^2$ for $0 \le \mu \le 2$ for a fixed $t \in [0, 2]$, i.e., the projection of the three-dimensional graph from Fig. 2 to the $(\mu, D^\mu t^2)$-plane.

The *Leibniz rule for fractional differentiation* is of primary importance for our considerations in this paper. It is well known [7], that if $f(\tau)$ is continuous in $[0, t]$ and $\varphi(\tau)$ has $n+1$ continuous derivatives in $[0, t]$, then the fractional derivative of the product $\varphi(t)f(t)$ is given by the Leibniz rule for fractional differentiation

$$D^\mu\left[\varphi(t)f(t)\right] = \sum_{k=0}^{m} \binom{\mu}{k} \varphi^{(k)}(t) D^{\mu-k} f(t) - R_m^\mu(t),$$

where $\mu > 0$, $m \ge \mu + 1$ and

$$R_m^\mu(t) = \frac{1}{m!\Gamma(-\mu)} \int_0^t (t-\tau)^{-\mu-1} f(\tau) \int_\tau^t \varphi^{(m+1)}(\xi)(t-\xi)^n \mathrm{d}\xi \mathrm{d}\tau.$$

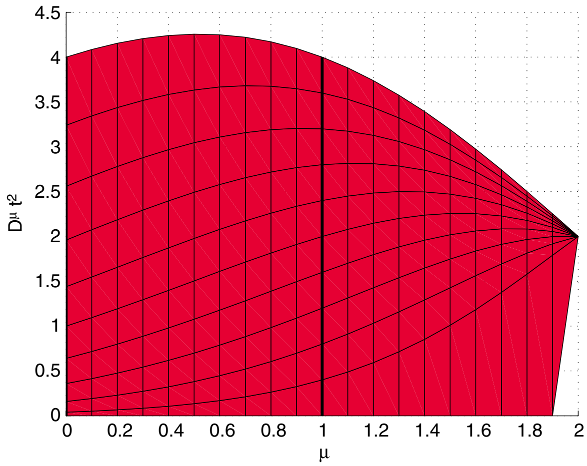

Fig. 3. $D^\mu t^2$ for $0 \le \mu \le 2$ for a fixed $t \in [0, 2]$ (step 0.2 for t).

If $\varphi(\tau)$ along with all its derivatives is continuous in $[0, t]$, the Leibniz rule takes the form

$$D^\mu\left[\varphi(t)f(t)\right] = \sum_{k=0}^{\infty} \binom{\mu}{k} \varphi^{(k)}(t) D^{\mu-k} f(t). \tag{2}$$

It is also useful to recall in this section that the ***Kummer differential equation***

$$xy'' + [c - x]y' - ay = 0, \tag{3}$$

has as a solution the ***Kummer function***

$${}_1F_1(a; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{x^k}{k!}, \tag{4}$$

where $(x)_n$ is the ***Pochhammer symbol***

$$(x)_n \equiv \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\dots(x+n-1).$$

2. Laguerre functions

The classical Laguerre polynomials $L_n^{(\alpha)}(x)$ are defined by Rodrigues' type formula

$$L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^x \frac{d^n}{dx^n}\left[x^{n+\alpha} e^{-x}\right], \tag{5}$$

where $n \in \mathbb{N}_0$ and $\alpha > -1$ [9]. Their basic properties [9,10] are given in Table 1. By taking the Riemann–Liouville fractional derivative D^ν in (5), we introduce functions that we naturally refer to as Laguerre functions.

Definition 1. The **Laguerre functions** are defined by the formula

$$L_\nu^{(\alpha)}(t) = \frac{1}{\Gamma(\nu+1)} e^t t^{-\alpha} D^\nu\left[e^{-t} t^{\nu+\alpha}\right], \tag{6}$$

where $\alpha > -1$ and $n - 1 < \nu < n$ $(n \in \mathbb{N})$.

Theorem 2. *If $\alpha > -1$, $n \in \mathbb{N}$, and $n - 1 < \nu < n$, the Laguerre functions are given by the following series* ***representation***

$$L_\nu^{(\alpha)}(t) = \sum_{k=0}^{\infty} \binom{\nu+\alpha}{\nu-k} \frac{(-t)^k}{k!}. \tag{7}$$

Proof. Since e^{-t} is continuously differentiable on $[0, t]$, the Leibniz rule (2), applied to the fractional derivative in (6), yields

$$L_\nu^{(\alpha)}(t) = \frac{1}{\Gamma(\nu+1)} e^t t^{-\alpha} \sum_{k=0}^{\infty} \binom{\nu}{k} \left\{D^k\left[e^{-t}\right]\right\} \left\{D^{\nu-k}\left[t^{\nu+\alpha}\right]\right\}.$$

Table 1
Properties of the classical Laguerre polynomials and the Laguerre functions ($n \in \mathbb{N}$, $n-1 \le \nu < n$).

Property	Laguerre polynomials	Laguerre functions
Definition	$L_n^{(\alpha)}(x) = \frac{1}{n!}x^{-\alpha}\mathrm{e}^x \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x^{n+\alpha}\mathrm{e}^{-x}\right]$	$L_\nu^{(\alpha)}(t) = \frac{1}{\Gamma(\nu+1)}\mathrm{e}^t t^{-\alpha} D^\nu\left[\mathrm{e}^{-t}t^{\nu+\alpha}\right]$
Explicitly	$L_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!}$	$L_\nu^{(\alpha)}(t) = \sum_{k=0}^{\infty}\binom{\nu+\alpha}{\nu-k}\frac{(-t)^k}{k!}$
Explicitly	$L_n^{(\alpha)}(x) = \binom{n+\alpha}{n}\,{}_1F_1(-n, \alpha+1; x)$	$L_\nu^{(\alpha)}(t) = \binom{\nu+\alpha}{\nu}\,{}_1F_1(-\nu; \alpha+1; t)$
Value at 0	$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$	$L_\nu^{(\alpha)}(0) = \binom{\nu+\alpha}{\nu}$
Recurrence	$L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) - L_{n-1}^{(\alpha+1)}(x)$	$L_\nu^{(\alpha)}(t) = L_\nu^{(\alpha+1)}(t) - L_{\nu-1}^{(\alpha+1)}(t)$
Recurrence	$\sum_{k=0}^{n} L_k^{(\alpha)}(x) = L_n^{(\alpha+1)}(x)$	$\sum_{k=0}^{m} L_{\nu+k}^{(\alpha)}(t) = L_{\nu+m}^{(\alpha+1)}(t) - L_{\nu-1}^{(\alpha+1)}(t)$, $(\alpha > 1)$
Derivative	$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L_n^{(\alpha)}(x) &= -L_{n-1}^{(\alpha+1)}(x)\\ &= x^{-1}\left[nL_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x)\right]\end{aligned}$	$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L_\nu^{(\alpha)}(t) &= -L_{\nu-1}^{(\alpha+1)}(t)\\ &= t^{-1}\left[\nu L_\nu^{(\alpha)}(t) - (\nu+\alpha)L_{\nu-1}^{(\alpha)}(t)\right]\end{aligned}$
Relation	$L_n^{(\alpha)}(x) = \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1-2\beta^{-1}x)$	$L_\nu^{(\alpha)}(t) = \lim_{\beta\to\infty} P_\nu^{(\alpha,\beta)}(1-2\beta^{-1}t)$
Equation	$xy'' + (\alpha+1-x)y' + ny = 0$	$xy'' + (\alpha+1-x)y' + \nu y = 0$

According to (1) we obtain

$$L_\nu^{(\alpha)}(t) = \frac{1}{\nu!}\mathrm{e}^t t^{-\alpha}\sum_{k=0}^{\infty}\frac{\nu!}{k!(\nu-k)!}(-1)^k\mathrm{e}^{-t}\frac{(\nu+\alpha)!}{(\nu+\alpha-\nu+k)!}t^{\alpha+k}.$$

Taking into account that the binomial coefficients with real arguments are defined by [8]

$$\binom{\alpha}{\beta} \equiv \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)\Gamma(1+\alpha-\beta)},$$

we see that

$$\begin{aligned}L_\nu^{(\alpha)}(t) &= \sum_{k=0}^{\infty}(-1)^k\frac{(\nu+\alpha)!}{(\nu-k)!(\alpha+k)!}\frac{t^k}{k!}\\ &= \sum_{k=0}^{\infty}\binom{\nu+\alpha}{\nu-k}\frac{(-t)^k}{k!},\end{aligned}$$

which proves the theorem. $\square$

Using (7) we could obtain another useful representation of the Laguerre functions.

Theorem 3. *If $\alpha > -1$, $n \in \mathbb{N}$ and $n-1 < \nu < n$, the Laguerre functions can be represented as*

$$L_\nu^{(\alpha)}(t) = \binom{\nu+\alpha}{\nu}\,{}_1F_1(-\nu; \alpha+1; t). \tag{8}$$

Proof. From (7) we have

$$\begin{aligned}L_\nu^{(\alpha)}(t) &= \sum_{k=0}^{\infty}\frac{(\nu+\alpha)!}{\nu!\alpha!}\frac{\nu!\alpha!}{(\nu-k)!(\alpha+k)!}\frac{(-t)^k}{k!}\\ &= \binom{\nu+\alpha}{\nu}\sum_{k=0}^{\infty}\frac{\nu.(\nu-1)\ldots(\nu-k+1)}{(\alpha+1)(\alpha+2)\ldots(\alpha+k)}\frac{(-t)^k}{k!}\\ &= \binom{\nu+\alpha}{\nu}\sum_{k=0}^{\infty}\frac{(-\nu).(-\nu+1)\ldots(-\nu+k-1)}{(\alpha+1)(\alpha+2)\ldots(\alpha+k)}\frac{t^k}{k!}.\end{aligned}$$

Applying now (4) we see indeed that

$$L_\nu^{(\alpha)}(t) = \binom{\nu+\alpha}{\nu}\,{}_1F_1(-\nu; \alpha+1; t). \quad \square$$

Theorems 2 and 3 together with some of the properties [11] of the Kummer function imply further interesting properties of the Laguerre functions. For example, by using (7), (8) and the differential rule for the Kummer function

$$D\,{}_1F_1(a, c; z) = \frac{a}{c}\,{}_1F_1(a+1, c+1; z),$$

it is not difficult to prove the following statement.

Theorem 4. *For every $n \in \mathbb{N}$ and $n-1 < \nu < n$, the Laguerre functions satisfy the following properties:*

(i) $\lim_{\nu\to n} L_\nu^{(\alpha)}(t) = L_n^{(\alpha)}(t)$;
(ii) $L_\nu^{(\alpha)}(0) = \binom{\nu+\alpha}{\nu}$;
(iii) $L_\nu^{(\alpha)}(t) = L_\nu^{(\alpha+1)}(t) - L_{\nu-1}^{(\alpha+1)}(t)$;
(iv) $\sum_{k=0}^{m} L_{\nu+k}^{(\alpha)}(t) = L_{\nu+m}^{(\alpha+1)}(t) - L_\nu^{(\alpha+1)}(t) + L_\nu^{(\alpha)}(t)$;
(v) *if further $\alpha > 1$:*

$$\sum_{k=0}^{m} L_{\nu+k}^{(\alpha)}(t) = L_{\nu+m}^{(\alpha+1)}(t) - L_{\nu-1}^{(\alpha+1)}(t);$$

(vi) $\frac{\mathrm{d}}{\mathrm{d}t} L_\nu^{(\alpha)}(t) = -L_{\nu-1}^{(\alpha+1)}(t) = t^{-1}\left[\nu L_\nu^{(\alpha)}(t) - (\nu+\alpha)L_{\nu-1}^{(\alpha)}(t)\right]$.

From Theorem 3, the representation of the Jacobi functions [12, p. 321]

$$P_\nu^{(\alpha,\beta)}(t) = \binom{\nu+\alpha}{\nu} {}_2F_1\left(-\nu, \nu+\alpha+\beta+1; \alpha+1; \frac{1-t}{2}\right)$$

in terms of the Gauss hypergeometric function ${}_2F_1(a, b; c; t)$ and the fact that [10, p. 102]

$${}_1F_1(\alpha; \gamma; t) = \lim_{\beta\to\infty} {}_2F_1(\alpha, \beta; \gamma; \beta^{-1}t),$$

it follows the validity of the final statement in this section.

Theorem 5. *The following relation between the Laguerre and Jacobi functions holds:*

$$L_\nu^{(\alpha)}(t) = \lim_{\beta\to\infty} P_\nu^{(\alpha,\beta)}(1-2\beta^{-1}t). \tag{9}$$

3. Generalized Kummer function

In this section we generalize (4) by solving the linear homogeneous fractional differential equation

$$t^\mu D^{2\mu} y(t) + [c - t^\mu] D^\mu y(t) - ay = 0, \quad 0 < \mu \le 1 \tag{10}$$

that we refer to as the **fractional Kummer differential equation**. It is clear that Eq. (10) is a generalization of the classical Kummer differential equation (3).

Definition 6. The **generalized Kummer function** is defined as

$${}_1^{\mu}F_1(a; c; t) = y_0 t^\rho \sum_{k=0}^{\infty} \prod_{j=0}^{k-1} \frac{g_j(\rho)}{f_{j+1}(\rho)} t^{k\mu}, \quad 0 < \mu \le 1, \tag{11}$$

where

$$f_k(\rho) \equiv \frac{\Gamma(1+\rho+k\mu)}{\Gamma(1+\rho+(k-2)\mu)} + c\frac{\Gamma(1+\rho+k\mu)}{\Gamma(1+\rho+(k-1)\mu)}, \tag{12}$$

$$g_k(\rho) \equiv \frac{\Gamma(1+\rho+k\mu)}{\Gamma(1+\rho+(k-1)\mu)} + a, \tag{13}$$

and $\rho > -1$ satisfies the equation

$$f_0(\rho) = \frac{\Gamma(1+\rho)}{\Gamma(1+\rho-2\mu)} + c\frac{\Gamma(1+\rho)}{\Gamma(1+\rho-\mu)} = 0. \tag{14}$$

Theorem 7. *The generalized Kummer function ${}_1^{\mu}F_1(a; c; t)$ is a solution of Eq. (10).*

Proof. Let us search for a solution of (10) in the form

$$y(t) = t^\rho \sum_{k=0}^{\infty} y_k t^{k\mu} = \sum_{k=0}^{\infty} y_k t^{\rho+k\mu}, \quad \rho > -1. \tag{15}$$

Inserting (15) into the Eq. (10) and using (1), we obtain

$$\begin{aligned}\mathbf{G}y &= t^{\mu}D^{2\mu}y + [c - t^{\mu}]\,D^{\mu}y - ay \\ &= \sum_{k=0}^{\infty} y_k \frac{\Gamma(1+\rho+k\mu)}{\Gamma(1+\rho+(k-2)\mu)} t^{\rho+(k-1)\mu} + c\sum_{k=0}^{\infty} y_k \frac{\Gamma(1+\rho+k\mu)}{\Gamma(1+\rho+(k-1)\mu)} t^{\rho+(k-1)\mu} \\ &\quad - \sum_{k=0}^{\infty} y_k \frac{\Gamma(1+\rho+k\mu)}{\Gamma(1+\rho+(k-1)\mu)} t^{\rho+k\mu} - a\sum_{k=0}^{\infty} y_k t^{\rho+k\mu}.\end{aligned}$$

Rearranging the terms in the sum, we get

$$\begin{aligned}\mathbf{G}y &= \sum_{k=0}^{\infty} y_k \left[\frac{\Gamma(1+\rho+k\mu)}{\Gamma(1+\rho+(k-2)\mu)} + c\frac{\Gamma(1+\rho+k\mu)}{\Gamma(1+\rho+(k-1)\mu)}\right] t^{\rho+(k-1)\mu} \\ &\quad - \sum_{k=0}^{\infty} y_k \left[\frac{\Gamma(1+\rho+k\mu)}{\Gamma(1+\rho+(k-1)\mu)} + a\right] t^{\rho+k\mu} \\ &= y_0 f_0(\rho) t^{\rho-\mu} + \sum_{k=0}^{\infty} [y_{k+1}f_{k+1} - y_k g_k]\, t^{\rho+k\mu},\end{aligned}$$

where f_k and g_k defined as in Eqs. (12) and (13), respectively.

Supposing that $y_0 \neq 0$, in order to get $y_0 f_0(\rho) = 0$, ρ has to be chosen such that (14) holds. Thus, with

$$y_{k+1} = \frac{g_k}{f_{k+1}} y_k = \prod_{j=0}^{k} \frac{g_j}{f_{j+1}} y_0$$

we see that

$$\mathbf{G}y(t) = 0. \quad \square$$

To justify the name generalized Kummer function, we outline that the Kummer function (4) is a special case of the fractional Kummer function (11) as $\mu = 1$, i.e.

$${}^{1}_{1}F_1(a; c; t) = {}_1F_1(a; c; t).$$

Finally we point out that the generalized Kummer function (11) is related to the fractional Gauss function (as defined in [12]) according to

$${}^{\mu}_{1}F_1(\alpha; \gamma; t) = \lim_{\beta\to\infty} {}^{\mu}_{2}F_1(\alpha, \beta; \gamma; \beta^{-1}t).$$

4. Generalized Laguerre function

In this section we follow the same approach as in Section 3 to generalize the Laguerre functions (6) by solving the linear homogeneous fractional differential equation

$$t^{\mu}D^{2\mu}y(t) + [\alpha + 1 - t^{\mu}]\,D^{\mu}y(t) + \nu y = 0, \quad 0 < \mu \le 1 \tag{16}$$

that we refer to as the **fractional Laguerre differential equation**.

Definition 8. The **generalized Laguerre function** is defined as

$${}^{\mu}L^{(\alpha)}_{\nu}(t) = y_0 t^{\rho} \sum_{k=0}^{\infty} \prod_{j=0}^{k-1} \frac{g^L_j(\rho)}{f^L_{j+1}(\rho)} t^{k\mu}, \quad 0 < \mu \le 1, \tag{17}$$

where

$$\begin{aligned}f^L_k(\rho) &\equiv \frac{\Gamma(1+\rho+k\mu)}{\Gamma(1+\rho+(k-2)\mu)} + (\alpha+1)\frac{\Gamma(1+\rho+k\mu)}{\Gamma(1+\rho+(k-1)\mu)}, \\ g_k(\rho) &\equiv \frac{\Gamma(1+\rho+k\mu)}{\Gamma(1+\rho+(k-1)\mu)} - \nu,\end{aligned}$$

and $\rho > -1$ satisfies the equation

$$f^L_0(\rho) = \frac{\Gamma(1+\rho)}{\Gamma(1+\rho-2\mu)} + (\alpha+1)\frac{\Gamma(1+\rho)}{\Gamma(1+\rho-\mu)} = 0.$$

Similarly to Theorem 7 we can prove the following statement.

Theorem 9. *The generalized Laguerre function* ${}^{\mu}L_{\nu}^{(\alpha)}(t)$ *is a solution of Eq.* (16).

From Theorem 9 it follows also that

$$ {}^{\mu}L_{\nu}^{(\alpha)}(t) = \binom{\nu+\alpha}{\nu}\,{}^{\mu}_{1}F_{1}\left(-\nu;\alpha+1;t\right). $$

Finally we outline that the Laguerre functions (6) are special cases of the generalized Laguerre function (17) as $\mu = 1$, i.e.

$$ {}^{1}L_{\nu}^{(\alpha)}(t) = L_{\nu}^{(\alpha)}(t). $$

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